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Weak Behaviour of Fourier–Jacobi Series

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Necessary conditions for the weak convergence of Fourier series in orthogonal polynomials are given. It is shown that the partial sum operator associated with the Jacobi series is of restricted weak type, but not of weak type, for the endpoints of the mean convergence interval.

INTRODUCTION

Let $d\alpha$ be a finite positive Borel measure on an interval $I \subset \mathbb{R}$ such that $\operatorname{supp}(d\alpha)$ is an infinite set and let $p_n(d\alpha)$ denote the corresponding orthonormal polynomials. For $f \in L^1(d\alpha)$, $S_n f$ stands for the nth partial sum of the orthogonal Fourier expansion of f in $\{p_n(d\alpha)\}_{n=0}^{\infty}$, that is,

$$S_n(f, x) = \sum_{k=0}^{n} a_k p_k(x), \qquad a_k = a_k(f) = \int_I f p_k \, d\alpha.$$

The study of the convergence of $S_n f$ in $L^p(d\alpha)$ $(p \neq 2)$ has been discussed for several classes of orthogonal polynomials (c.f. Askey and Wainger [1], Badkov [2–4], Muckenhoupt [11–13], Newman and Rudin [16], Pollard [17–19], and Wing [24]). For instance, in the case of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ which are orthogonal in $[-1, 1]$ with respect to the weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta \geqslant -1/2$, Pollard proved that $|1/p - 1/2| < \min\{1/(4\alpha+4), 1/(4\beta+4)\}$ is a sufficient condition for uniform boundedness $\|S_n f\|_{p,w} \leqslant C\|f\|_{p,w}$, which is equivalent to convergence in $L^p(w)$, $1 < p < \infty$. Newman and Rudin showed that the previous condition is also necessary and later Muckenhoupt extended these results to $\alpha, \beta > -1$.

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The aim of this paper is to examine the weak behaviour of the Fourier–Jacobi expansion, that is, to study if there exists a constant C, independent of n, y and f, such that

$$\int_{|S_n(f,x)|>y} w(x)\,dx \leqslant Cy^{-p}\int_{-1}^{1} |f(x)|^p\, w(x)\,dx, \qquad y>0,$$

i.e., if S_n is uniformly bounded from $L^p(w)$ into $L^p_*(w)$, $1<p<\infty$.

By using interpolation [22, Theorem 3.15, p. 197], the range of p's for which there exists convergence of $S_n f$ in $L^p(w)$ is always an interval, named the mean convergence interval. Moreover, the previous weak inequality only can be true, besides the mean convergence interval, in its endpoints. Since for $-1<\alpha$, $\beta\leqslant -1/2$, the condition $|1/p-1/2|<\min\{1/(4\alpha+4), 1/(4\beta+4)\}$ is trivial for $p\in(1,\infty)$, we suppose, by symmetry, $\alpha\geqslant\beta$ and $\alpha>-1/2$. Then the mean convergence interval is $4(\alpha+1)/(2\alpha+3)<p<4(\alpha+1)/(2\alpha+1)$. For the Fourier–Legendre expansion ($\alpha=\beta=0$) and $p=4$ Chanillo [5] proved that the partial sum operator is not of weak type (4, 4), but is of restricted weak type (4, 4) (and (4/3, 4/3), by duality), i.e., it is weakly bounded on characteristic functions.

On the other hand, Máté, Nevai, and Totik [10] obtained, in a general way, necessary conditions for the mean convergence of Fourier expansions.

THEOREM (Máté–Nevai–Totik). *Let $d\alpha$ be such that* $\operatorname{supp}(d\alpha)=[-1,1]$, $\alpha'>0$ *almost everywhere, U and V nonnegative Borel measurable functions such that neither of them vanishes almost everywhere in* $[-1,1]$ *and V is finite on a set with positive Lebesgue measure. If S_n is uniformly bounded from $L^p(V(x)^p\,d\alpha)$ into $L^p(U(x)^p\,d\alpha)$, then*

(i) $U(x)^p\in L^1(d\alpha)$, $V(x)^{-q}\in L^1(d\alpha)$, $q=p/(p-1)$

(ii) $\displaystyle\int_{-1}^{1} U(x)^p\,\alpha'(x)^{1-p/2}\,(1-x^2)^{-p/4}\,dx<\infty$

(iii) $\displaystyle\int_{-1}^{1} V(x)^{-q}\,\alpha'(x)^{1-q/2}\,(1-x^2)^{-q/4}\,dx<\infty.$

When $d\alpha$ and $d\beta=U(x)^p\,d\alpha=V(x)^p\,d\alpha$ are generalized Jacobi measures, these conditions turn out to be sufficient too [2–4].

This paper is organized as follows. In Section 1 we obtain necessary conditions for weak convergence. These allow us to prove that S_n is not of weak type (p,p) for $p=4(\alpha+1)/(2\alpha+3)$. From these conditions it follows that (i), (ii), and (iii) are necessary not only for mean convergence but also for weak convergence. We end this section by giving an example which shows that they are not sufficient. In Section 2 we prove, by using similar arguments to [5], that S_n is not of weak type for $p=4(\alpha+1)/(2\alpha+1)$.

Finally, in Section 3, we obtain that the partial sum operator is of restricted weak type (p, p) for both endpoints of the mean convergence interval when $\alpha, \beta \geqslant -1/2$ and one of them is bigger than $-1/2$.

SECTION 1

Assume $\operatorname{supp}(d\alpha) = [-1, 1]$, $\alpha' > 0$ a.e., and let $\{p_n(x)\}_{n=0}^{\infty}$ be the corresponding orthonormal polynomials. C is used to denote positive constants not necessarily the same in each occurrence and $q = p/(p-1)$. For $f \in L^1(d\alpha)$, let $S_n f$ denote the nth partial sum of the orthogonal Fourier expansion of f in $\{p_n(x)\}_{n=0}^{\infty}$. We want to find necessary conditions for the weak convergence of S_n.

LEMMA 1. *Let U and V be weights and let $1 < p < \infty$. If there exists a constant C such that for every $f \in L^p(V^p d\alpha)$ the inequality*

$$\|S_n f\|_{L^p_*(U^p d\alpha)} \leqslant C \|f\|_{L^p(V^p d\alpha)} \tag{1.1}$$

holds for all integers $n \geqslant 0$, then

$$\|p_n\|_{L^q(V^{-q} d\alpha)} \|p_n\|_{L^p_*(U^p d\alpha)} \leqslant C. \tag{1.2}$$

Proof. It follows from (1.1) that

$$\|a_n(f)\, p_n\|_{L^p_*(U^p d\alpha)} = \|S_n f - S_{n-1} f\|_{L^p_*(U^p d\alpha)} \leqslant C \|f\|_{L^p(V^p d\alpha)}$$

Thus,

$$|a_n(f)| \leqslant C(\|p_n\|_{L^p_*(U^p d\alpha)})^{-1}\, \|f\|_{L^p(V^p d\alpha)}$$

Therefore, every operator

$$p_n/V^p \colon L^p(V^p d\alpha) \to \mathbb{R}$$

$$f \to \left(\int_{-1}^{1} (p_n/V^p) f V^p d\alpha\right) = a_n(f)$$

is bounded and, by duality, its norm as operator coincides with the norm as function in $L^q(V^p d\alpha)$. Thus

$$\|p_n/V^p\|_{L^q(V^p d\alpha)} \leqslant C(\|p_n\|_{L^p_*(U^p d\alpha)})^{-1}$$

and (1.2) indeed holds.

In order to prove the main theorem, we will use the following result established by Máté, Nevai, and Totik [10, Theorem 2].

LEMMA 2. *Let* $\operatorname{supp}(d\alpha)=[-1,1]$, $\alpha'>0$ *a.e. in* $[-1,1]$, *and* $0<p\leqslant\infty$. *There exists a constant C such that if g is a Lebesgue-measurable function in* $[-1,1]$, *then*

$$\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\|_{L^p(|g|^p\,dx)}\leqslant C\liminf_{n\to\infty}\|p_n\|_{L^p(|g|^p\,dx)}.$$

In particular, if

$$\liminf_{n\to\infty}\|p_n\|_{L^p(|g|^p\,dx)}=0$$

then $g=0$ *a.e.*

THEOREM 1. *Let* $d\alpha$, U, *and* V *be as in Lemma* 1. *If there exists a constant C such that*

$$\|S_nf\|_{L^p_*(U^p d\alpha)}\leqslant C\|f\|_{L^p(V^p d\alpha)}$$

holds for all integers $n\geqslant 0$ *and every* $f\in L^p(V^p d\alpha)$, *then*

$$U^p,\ V^{-q}\in L^1(d\alpha) \tag{1.3}$$

$$\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\in L^p_*(U^p\alpha'dx) \tag{1.4}$$

$$\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\in L^q(V^{-q}\alpha'dx). \tag{1.5}$$

Proof. Taking $n=0$ in Lemma 1 we obtain (1.3).

In order to prove (1.4) and (1.5), we use the result

$$\|f\|_{L^p_*(dm)}\leqslant\sup_E\frac{\|f\aleph_E\|_r}{\|\aleph_E\|_s}\leqslant\left(\frac{p}{p-r}\right)^{1/r}\|f\|_{L^p_*(dm)},$$

where dm is a Borel measure, $0<r<p<\infty$, $1/s=1/r-1/p$, and the supremum is taken over all measurable sets E such that $0<m(E)<\infty$ [7, Lemma V.2.8, p. 485]. From Lemma 2 and this inequality, it follows that

$$\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\|_{L^p_*(|g|^p dx)}\leqslant C\liminf_{n\to\infty}\|p_n\|_{L^p_*(|g|^p dx)}.$$

Now, taking $\liminf_{n\to\infty}$ in (1.2), we obtain

$$\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\|_{L^q(V^{-q}\alpha'dx)}\,\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\|_{L^p_*(U^p\alpha'dx)}\leqslant C.$$

As none of these norms can vanish, we get (1.4) and (1.5).

An easy consequence from Theorem 1 is

COROLLARY 1. *If $d\alpha$, U, and V are as in Lemma* 1 *and if S_n is uniformly bounded from $L^p(V(x)^p\,d\alpha)$ into $L^p_*(U(x)^p\,d\alpha)$ and from $L^q(U(x)^{-q}\,d\alpha)$ into $L^q_*(V(x)^{-q}\,d\alpha)$, then we have*

$$U(x)^p \in L^1(d\alpha), \qquad V(x)^{-q} \in L^1(d\alpha) \tag{1.6}$$

$$\int_{-1}^{1} V(x)^{-q}\,\alpha'(x)^{1-q/2}\,(1-x^2)^{-q/4}\,dx < \infty \tag{1.7}$$

$$\int_{-1}^{1} U(x)^p\,\alpha'(x)^{1-p/2}\,(1-x^2)^{-p/4}\,dx < \infty. \tag{1.8}$$

Remark 1. Let S_n denote the nth partial sum of the Fourier–Jacobi expansion ($\alpha'(x)=(1-x)^\alpha\,(1+x)^\beta$, $\alpha, \beta \geqslant -1/2, \alpha > -1/2$) and $U(x)=V(x)=1$. Because of (1.3) and (1.5) in Theorem 1 we obtain the conditions

$$|A+1|\;|1/p-1/2| < (A+1)/2, \qquad (A+1)(1/p-1/2) < 1/4,$$

where A is α or β, the same in each statement. As the latter inequality is not satisfied for $p=4(\alpha+1)/(2\alpha+3)$, it implies that S_n is not of weak type (p, p) for the lower endpoint of the interval of mean convergence. The same happens with generalized Jacobi polynomials.

Remark 2. The conditions (1.6), (1.7), and (1.8) are the same as (i), (ii), and (iii) in the Introduction. These are necessary conditions for the boundedness of S_n from $L^p(V(x)^p\,d\alpha)$ into $L^p(U(x)^p\,d\alpha)$ or equivalently from $L^q(U(x)^{-q}\,d\alpha)$ into $L^q(V(x)^{-q}\,d\alpha)$. This points out that the conditions obtained by Máté, Nevai, and Totik are necessary not only for the mean convergence but also for the weak convergence.

Remark 3. Let us prove that Máté–Nevai–Totik conditions are not sufficient for weak convergence. Consider the Fourier–Legendre expansion ($d\alpha=dx$), $p=4$, and take

$$U(x)=\left|\log\left(\frac{1+x}{4}\right)\right|^{-5/8}\left|\log\left(\frac{1-x}{4}\right)\right|^{-5/8},$$

$$V(x)=\left|\log\left(\frac{1+x}{4}\right)\right|^{-3/8}\left|\log\left(\frac{1-x}{4}\right)\right|^{-3/8}.$$

It is immediate that U and V satisfy (1.6) in Corollary 1. In order to prove the remaining conditions, (1.7) and (1.8), we will show that the weights U and V satisfy even stronger ones, that is

$$((1-x^2)^\delta\,U(x)^{4\delta}, (1-x^2)^\delta\,V(x)^{4\delta}) \in A_4(-1, 1) \quad \text{for some } \delta > 1, \tag{1.9}$$

$$((1-x^2)^{-1}\,U(x)^4, (1-x^2)^{-1}\,V(x)^4) \in A_4(-1, 1), \tag{1.10}$$

where $A_p(-1,1)$ stands for the Muckenhoupt A_p classes [9, 14], i.e., $(u, v) \in A_p(-1,1)$, $1<p<\infty$, if

$$\int_I u(x)\,dx \left(\int_I v(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant C|I|^p$$

for every interval $I \subset [-1,1]$, and $|I|$ denotes the Lebesgue measure of the interval I. The weight w belongs to A_1 if $Mw(x) \leqslant Cw(x)$ a.e., where M denotes the Hardy–Littlewood function.

In order to prove (1.9), by using symmetry and change of variable, it suffices to show that

$$(u, v)=(x^{\delta}\,|\log x|^{-5\delta/2}, x^{\delta}\,|\log x|^{-3\delta/2}) \in A_4(0, 1/2).$$

Let

$$w(x)=x\,|\log x|^{-2}=x^{-1/2}(x^{-1/2}|\log x|^{2/3})^{-3}=w_1(x)\,w_2(x)^{-3}.$$

It is known that $w_1 \in A_1$ and it is not difficult to prove that $w_2 \in A_1$. Thus, $w \in A_4$ by using the factorization theorem for A_p weights [6]. Now, (1.9) follows from [15, Theorem 2] and it implies that the Hilbert transform H is bounded from $L^p(v)$ into $L^p(u)$, which will be used later.

On the other hand, if $u(x)=x^{-1}|\log x|^b$ and $v(x)=x^{-1}|\log x|^B$, it can be shown that $(u, v) \in A_p(0, 1/2)$ if and only if $-b>1$ and $b+1 \leqslant B$. Then (1.10) follows obviously.

If $p_n(x)$ stands for the Legendre orthonormal polynomials, the partial sum operator can be decomposed [17] as

$$\begin{aligned} S_n(f, x)=\alpha_n p_{n+1}(x) &\int_{-1}^{1} \frac{p_n(t)-p_{n+2}(t)}{x-t} f(t)\,dt \\ &+\alpha_n(p_{n+2}(x)-p_n(x)) \int_{-1}^{1} \frac{p_{n+1}(t)}{x-t} f(t)\,dt \\ &-\beta_n p_{n+1}(x) \int_{-1}^{1} p_{n+1}(t) f(t)\,dt, \end{aligned} \tag{1.11}$$

where α_n and β_n are bounded and

$$(1-x^2)^{1/4}\,|p_n(x)| \leqslant C, \qquad (1-x^2)^{-1/4}\,|p_n(x)-p_{n+2}(x)| \leqslant C. \tag{1.12}$$

We now try to estimate the three summands of Eq. (1.11). We begin by estimating the last term using Hölder's inequality, the first part of (1.12), and (1.10). We have

$$\int_{-1}^{1} |p_{n+1}(t) f(t)|\, dt \leqslant C\left(\int_{-1}^{1} (1-t^2)^{-1/3}\, V(t)^{-4/3}\, dt\right)^{3/4} \times\left(\int_{-1}^{1} |f(t)\, V(t)|^4\, dt\right)^{1/4} = C_1 \|fV\|_4.$$

Therefore

$$\int_{-1}^{1} \left|p_{n+1}(x) \int_{-1}^{1} p_{n+1}(t) f(t)\, dt\right|^4 U(x)^4\, dx \leqslant C \int_{-1}^{1} (1-x^2)^{-1}\, U(x)^4\, dx \int_{-1}^{1} |f(t)\, V(t)|^4\, dt = C_1 \|fV\|_4^4.$$

We now estimate the middle term using (1.9) and (1.12):

$$\int_{-1}^{1} \left|(p_{n+2}(x) - p_n(x)) \int_{-1}^{1} \frac{p_{n+1}(t) f(t)}{x-t}\, dt\right|^4 U(x)^4\, dx \leqslant C \int_{-1}^{1} |H(p_{n+1} f, x)|^4\, (1-x^2)\, U(x)^4\, dx \leqslant C_1 \int_{-1}^{1} |p_{n+1}(x) f(x)|^4\, (1-x^2)\, V(x)^4\, dx \leqslant C_2 \|fV\|_4^4.$$

Finally, we will prove that the first term is not weakly bounded. The proof is by contradiction. Let us assume that there exists a constant C, independent of n and $f \in L^4(V^4)$, such that

$$\int_{|p_{n+1}(x)|\, |H((p_n - p_{n-2}) f, x)| > y} U(x)^4\, dx \leqslant C y^{-4}\, \|fV\|_4^4.$$

Then it will be enough to construct a sequence of functions $\{f_n(t)\}$ such that the constant appearing in the above inequality grows with n. A slight modification of the argument used by Chanillo proves that $C \geqslant (\log n)^{3/2}$. Therefore, the partial sum operator is not of weak type (4, 4).

SECTION 2

Let S_n denote the nth partial sum of the Fourier–Jacobi expansion with respect to $w(x) = (1-x)^\alpha\, (1+x)^\beta$, with $\alpha \geqslant \beta$ and $\alpha > -1/2$. Then the interval of mean convergence is given by $4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$. Theorem 1 works to prove that S_n is not of weak type on $L^p(w)$ for $p = 4(\alpha+1)/(2\alpha+3)$, since (1.5) is not satisfied. But it is not useful to show that S_n is not of weak type for $p = 4(\alpha+1)/(2\alpha+1)$. It leads us to make use of other arguments.

THEOREM 2. *Let $r = 4(\alpha+1)/(2\alpha+1)$. Then there exists no constant C, independent of n and $f \in L^r(w)$, such that*

$$\|S_n f\|_{L^r_*(w)} \leqslant C\|f\|_{L^r(w)}. \tag{2.1}$$

Proof. In what follows assume $\operatorname{supp}(f) \subset [0, 1]$. Let $p_n(x)$ denote the orthonormal polynomials with respect to $w(x)$ and $q_n(x)$ the orthonormal polynomials with respect to $w(x)(1-x^2)$. Moreover, if (2.1) is true, the same happens if the left integral is only taken over the set $\{x \in (0, 1), |S_n(f, x)| > y\}$. Then, we will suppose all integrals are restricted by the condition $x \in (0, 1)$.

Pollard [18] proved that in

$$S_n(f, x) = \int_{-1}^{1} f(t)\, K_n(x, t)\, w(t)\, dt$$

the kernel $K_n(x, t)$ can be decomposed in the form

$$K_n(x, t) = r_n T_1(n, x, t) + s_n T_2(n, x, t) + s_n T_3(n, x, t), \tag{2.2}$$

where r_n and s_n are bounded and

$$T_1(n, x, t) = p_n(x)\, p_n(t) \tag{2.3}$$

$$T_2(n, x, t) = (1-t^2)\frac{p_n(x)\, q_{n-1}(t)}{x-t} \tag{2.4}$$

$$T_3(n, x, t) = T_2(n, t, x) = (1-x^2)\frac{p_n(t)\, q_{n-1}(x)}{t-x}. \tag{2.5}$$

As $\alpha \geqslant -1/2$, from [23, p. 169], if $x \in [0, 1]$, we have the estimates

$$|p_n(x)| \leqslant C(1-x)^{-\alpha/2-1/4}, \tag{2.6}$$

$$|q_n(x)| \leqslant C(1-x)^{-\alpha/2-3/4}. \tag{2.7}$$

Let

$$W_i(f, x) = W_{i,n}(f, x) = \int_{-1}^{1} f(t)\, T_i(n, x, t)\, w(t)\, dt \qquad (i = 1, 2, 3).$$

We try to estimate the three terms

$$\int_{|W_i(f,x)|>y} w(x)\,dx \qquad (i=1,2,3).$$

$(i=1)$ By using (2.6) and Hölder's inequality, we have

$$\begin{aligned} |W_1(f,x)| &= |p_n(x)\int_0^1 f(t)\,p_n(t)\,w(t)\,dt| \\ &\leqslant C(1-x)^{-\alpha/2-1/4}\left(\int_0^1 |f(t)|^p\,w(t)\,dt\right)^{1/p} \\ &\quad\times\left(\int_0^1 (1-t)^{q(-\alpha/2-1/4)}\,w(t)\,dt\right)^{1/q} \\ &= C_1(1-x)^{-\alpha/2-1/4}\,\|f\|_{p,w} \end{aligned}$$

if $q(-\alpha/2-1/4)+\alpha>-1$. Applying this to $p=r$ we get

$$\int_{|W_1(f,x)|>y} w(x)\,dx \leqslant \int_{(1-x)^{-1/4-\alpha/2}>y/(C\|f\|_{r,w})} w(x)\,dx \leqslant C_1\,y^{-r}\|f\|_{r,w}^r,$$

which shows that W_1 is weakly bounded.

$(i=3)$ Let H denote the Hilbert transform. It is well known that H is bounded from $L^p(u)$ into $L^p(u)$ if and only if the weight u belongs to A_p [9]. By using (2.6) and (2.7), we get

$$\begin{aligned} &\int_0^1 |W_3(f,x)|^p\,w(x)\,dx \\ &\quad\leqslant C\int_0^1 |H(f(t)\,p_n(t)\,w(t),x)|^p\,(1-x)^{(1/4-\alpha/2)p+\alpha}\,dx. \end{aligned} \tag{2.8}$$

Recalling that $(1-x)^{(1/4-\alpha/2)p+\alpha}\in A_p(0,1)$ iff $-1<(1/4-\alpha/2)p+\alpha<p-1$, [21], and it is verified for $p=r$; then (2.8) and (2.6) yield

$$\begin{aligned} &\int_0^1 |W_3(f,x)|^r\,w(x)\,dx \\ &\quad\leqslant C\int_0^1 |f(x)\,p_n(x)\,w(x)|^r\,(1-x)^{(1/4-\alpha/2)r+\alpha}\,dx \leqslant C_1\,\|f\|_{r,w}^r \end{aligned}$$

and therefore W_3 is strongly bounded.

($i=2$) We shall prove that there is no constant C, independent of n and f, such that

$$\int_{|p_n(x)\,H(f(t)\,q_{n-1}(t)\,(1-t^2)\,w(t),\,x)|>y} w(x)\,dx \leqslant C y^{-r}\,\|f\|^r_{r,w} \tag{2.9}$$

The proof is by contradiction, constructing a sequence of functions $\{f_{m,n}(t)\}$ such that the constant C appearing in the previous inequality grows with m. In order to get it we try to remove the term $(x-t)^{-1}$ in the Hilbert transform. We need sharper forms of (2.6) and (2.7).

Because of [23, Theorem 8.21.13], if $N=n+(\alpha+\beta+1)/2$ and $\gamma=-(\alpha+1/2)\,\pi/2$, we have

$$\begin{aligned} p_n(\cos\theta) &= (2^{\alpha+\beta}\pi)^{-1/2}\,(\sin(\theta/2))^{-\alpha-1/2}\,(\cos(\theta/2))^{-\beta-1/2} \\ &\quad\times[\cos(N\theta+\gamma)+(n\sin\theta)^{-1}\,O(1)], \end{aligned}$$

where $c/n\leqslant\theta\leqslant\pi-c/n$, c being a fixed positive number.

We will restrict our attention to $\theta\leqslant\pi/2$ and choose M large enough and such that $M-\alpha/2$ is a positive integer. If $M\pi/n\leqslant\theta\leqslant(M+1/8)\,\pi/n$, we have

$$\begin{aligned} N\theta+\gamma &\leqslant \left(n+\frac{\alpha+\beta+1}{2}\right)\left(M+\frac{1}{8}\right)\frac{\pi}{n} \\ &\quad-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2} \xrightarrow[n\to\infty]{} \left(M-\frac{\alpha}{2}\right)\pi-\frac{\pi}{8} \end{aligned}$$

and

$$N\theta+\gamma \geqslant \left(n+\frac{\alpha+\beta+1}{2}\right)\frac{M\pi}{n}-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2} \xrightarrow[n\to\infty]{} \left(M-\frac{\alpha}{2}\right)\pi-\frac{\pi}{4}.$$

Hence, for every $\varepsilon>0$, there exists n_0 such that

$$\left(M-\frac{\alpha}{2}\right)\pi-\frac{\pi}{4}-\varepsilon\leqslant N\theta+\gamma\leqslant\left(M-\frac{\alpha}{2}\right)\pi-\frac{\pi}{8}+\varepsilon, \qquad n\geqslant n_0.$$

Therefore $|\cos(N\theta+\gamma)|$ is bounded below by a positive constant for n large enough, and the same happens to $|\cos(N\theta+\gamma)|-|(n\sin\theta)^{-1}\,O(1)|$. Then $p_n(\cos\theta)\geqslant C(\sin(\theta/2))^{-\alpha-1/2}$, and taking $x=\cos\theta$ we get

$$|p_n(x)|\geqslant C(1-x^2)^{-\alpha/2-1/4}\geqslant C_1(1-x)^{-\alpha/2-1/4}\geqslant C_2 n^{\alpha+1/2} \tag{2.10}$$

for $n\geqslant n_0$ and $x\in[\cos(M+1/8)\,\pi/n,\ \cos M\pi/n]=I_n$.

We define

$$f_{m,n}(t)=\frac{\operatorname{sgn} q_{n-1}(t)}{(1-t)^{\alpha/2+1/4}} \qquad \text{if} \quad 0\leqslant t\leqslant \cos\left(\frac{2M\pi}{m}\right),\ m<n$$

and $f_{m,n}(t)=0$ elsewhere.

Now we are going to estimate the left side in (2.9). If $x\in I_n$ and $t\in \operatorname{supp}(f_{m,n})$ then $0<x-t<1-t$. With the aid of Lemma 2, there exists a subsequence of n's for which

$$\begin{aligned}|H(f_{m,n}(t)\, q_{n-1}(t)\,(1-t)^{\alpha+1}\,(1+t)^{\beta+1}, x)|\\ =\left|\int_0^{\cos(2M\pi/m)} \frac{|q_{n-1}(t)|\,(1-t)^{\alpha+1}\,(1+t)^{\beta+1}}{(1-t)^{\alpha/2+1/4}\,(x-t)}\, dt\right| \geqslant C\log m. \qquad (2.11)\end{aligned}$$

Thus, this and (2.10) imply

$$\begin{aligned}|p_n(x)|\,|H(f_{m,n}(t)\, q_{n-1}(t)\,(1-t)^{\alpha+1}\,(1+t)^{\beta+1}, x)|\\ \geqslant Cn^{\alpha+1/2}\log m, \qquad x\in I_n. \qquad (2.12)\end{aligned}$$

On the other hand, it is easy to check that

$$\int_0^1 |f_{m,n}(t)|^r\,(1-t)^{\alpha}\,(1+t)^{\beta}\, dt\leqslant C\int_0^{\cos(2M\pi/m)} (1-t)^{-1}\, dt\leqslant C_1\log m. \quad (2.13)$$

Finally, assume (2.9). By using (2.13) it follows that

$$\begin{aligned}y^{-r}\log n &\geqslant Cy^{-r}\int_0^1 |f_{m,n}(t)|^r\, w(t)\, dt\\ &\geqslant C_1\int_{|p_n(x)\, H(f_{m,n}(t)\, q_{n-1}(t)\,(1-t^2)\, w(t),\, x)|>y} w(x)\, dx. \qquad (2.14)\end{aligned}$$

Choose $y=Cn^{\alpha+1/2}\log m$. As $|I_n|=O(n^{-2})$, using (2.12) and (2.14) we have

$$\begin{aligned}&Cn^{-2-2\alpha}(\log m)^{-(2\alpha+3)/(2\alpha+1)}\\ &\quad\geqslant C_1\int_{I_n} (1-x)^{\alpha}\,(1+x)^{\beta}\, dx\geqslant C_2\,|I_n|\, n^{-2\alpha}\geqslant C_3 n^{-2-2\alpha},\end{aligned}$$

i.e., $(\log m)^{-(2\alpha+3)/(2\alpha+1)}\geqslant C$, which is absurd.

SECTION 3

The aim of this part is to examine the restricted weak behaviour of the nth partial sum of the Fourier–Jacobi expansion. Assume that

$w(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha \geqslant \beta \geqslant -1/2$ and $\alpha > -1/2$, and so the interval of mean convergence turns out to be $4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$. We need some notations to establish our result. The Lebesgue measure of any set $E \subset (-1, 1)$ will be denoted $|E|$, and $\aleph_E(x)$ will as usual denote the characteristic function of the set E. The reader is refered to [8] or [22] for notations and results about Lorentz spaces.

THEOREM 3. *Let* $p = 4(\alpha+1)/(2\alpha+1)$ *or* $p = 4(\alpha+1)/(2\alpha+3)$. *Then there exists a constant* C, *independent of* n *and* $E \subset (-1, 1)$, *such that*

$$\int_{|S_n(\aleph_E, x)| > y} w(x)\, dx \leqslant C y^{-p} \int_E w(x)\, dx.$$

Proof. By standard duality arguments it is enough to prove this for $p = 4(\alpha+1)/(2\alpha+1)$. Since $\beta \geqslant -1/2$, we have estimates analogous to (2.6) and (2.7) in the interval $(-1, 0)$,

$$|p_n(x)| \leqslant C(1-x^2)^{-1/4} w(x)^{-1/2},$$
$$|q_n(x)| \leqslant C(1-x^2)^{-3/4} w(x)^{-1/2}, \qquad x \in (-1, 1). \tag{3.1}$$

We proceed as in Section 2 by making the same decomposition for the kernel in $S_n(\aleph_E, x)$. We can see that W_1 and W_3 are weakly bounded. Therefore we only need to prove that

$$\int_{|\int_{-1}^1 \aleph_E(t)\, T_2(n, x, t)\, w(t)\, dt| > y} w(x)\, dx \leqslant C y^{-p} \int_E w(x)\, dx$$

for every measurable set $E \subset (-1, 1)$. By (3.1),

$$\left| \int_{-1}^1 \aleph_E(t)\, T_2(n, x, t)\, w(t)\, dt \right|$$
$$\leqslant C(1-x^2)^{-1/4} w(x)^{-1/2} \, |H(\aleph_E(t)\, q_{n-1}(t)\, (1-t^2)\, w(t), x)|.$$

Therefore, if we denote

$$U = U(E, y, n)$$
$$= \{x \in (-1, 1) : (1-x^2)^{-1/4}$$
$$\times w(x)^{-1/2} \, |H(\aleph_E(t)\, q_{n-1}(t)\, (1-t^2)\, w(t), x)| > y\}$$

it suffices to prove

$$y^p \int_U w(x)\, dx \leqslant C \int_E w(x)\, dx.$$

We can write a similar proof to that of [5]. Decompose $E=E_1\cup E_2$, $E_1=E\cap[0,1)$, $E_2=E\cap(-1,0)$, and let $U_1=U(E_1,y,n)$, $U_2=U(E_2,y,n)$, Since in $|x|\leqslant 3/4$ both $(1-x)$ and $(1+x)$ have lower and upper positive bounds and since the Hilbert transform is a bounded operator in $L^p(dx)$, $1<p<\infty$ (M. Riesz's Theorem [20]), the following inequality holds:

$$y^p\int_{U\cap\{|x|\leqslant 3/4\}} w(x)\,dx\leqslant C\int_E w(x)\,dx.$$

Hence, we must show that

$$y^p\int_{U_i\cap\{3/4<|x|<1\}} w(x)\,dx\leqslant C\int_E w(x)\,dx,\qquad i=1,2. \tag{3.2}$$

We only prove (3.2) for $i=1$ since the estimates for $i=2$ are made in similar way. We begin considering $x\in(-1,-3/4)$. As $E_1\subset[0,1]$, the term $(x-t)^{-1}$ in the Hilbert transform can be dropped. By using (3.1) and Hölder's inequality for $p=4(\alpha+1)/(2\alpha+1)$ and $q=4(\alpha+1)/(2\alpha+3)$, we easily get

$$|H(\aleph_{E_1}(t)\,q_{n-1}(t)\,(1-t^2)\,w(t),x)|\leqslant C\left(\int_{E_1} w(t)\,dt\right)^{1/p}$$

Then, if $\alpha=\beta$, by using (3.1) again, we have

$$y^p\int_{U_1\cap\{-1<x<-3/4\}} w(x)\,dx\leqslant Cy^p\int_B (1+x)^\alpha\,dx,$$

where B is the set

$$\left\{x\in(-1,-3/4):C(1+x)^{-\alpha/2-1/4}\left(\int_{E_1} w(t)\,dt\right)^{1/p}>y\right\}.$$

Then

$$y^p\int_{U_1\cap\{-1<x<-3/4\}} w(x)\,dx\leqslant C\int_{E_1} w(x)\,dx$$

is obtained straightforwardly.

The previous inequality is also true when $\beta<\alpha$ and it can be easily shown, taking into account that the weak norm is smaller than the strong norm.

Consequently (3.2) will be proved if we show

$$y^p \int_{U_1 \cap \{3/4 < x < 1\}} w(x)\, dx \leqslant C \int_E w(x)\, dx. \tag{3.3}$$

Let us define the sets

$$A = U_1 \cap \{3/4 < x < 1\},$$
$$I_k = \{x \in (0, 1): 2^{-k-1} \leqslant 1 - x < 2^{-k}\},$$
$$A_k = A \cap I_k, \qquad k \geqslant 2.$$

We have

$$A = \bigcup_{k=2}^{\infty} A_k, \qquad A_k \cap A_j = \varnothing \ \forall k \neq j, \qquad \bigcup_{k=2}^{\infty} I_k = (3/4, 1).$$

Let us also set for $k \geqslant 2$

$$E_{11}^{(k)} = E_1 \cap [0, 1 - 2^{-k+1})$$
$$E_{12}^{(k)} = E_1 \cap [1 - 2^{-k+1}, 1 - 2^{-k-2})$$
$$E_{13}^{(k)} = E_1 \cap [1 - 2^{-k-2}, 1).$$

For every $k \geqslant 2$, $E_{1m}^{(k)}$ $(m = 1, 2, 3)$ are non-intersecting sets whose union is E_1. If we denote by $\aleph_m^{(k)}$ the characteristic function of the set $E_{1m}^{(k)}$ $(m = 1, 2, 3, k \geqslant 2)$, then

$$\aleph_{E_1} = \aleph_1^{(k)} + \aleph_2^{(k)} + \aleph_3^{(k)}, \qquad k \geqslant 2.$$

Again for $k \geqslant 2$ and $m = 1, 2, 3$, let

$$A_k^{(m)} = \{x \in I_k : |(1 - x^2)^{-1/4} w(x)^{-1/2} H(\aleph_m^{(k)}(t) q_{n-1}(t)(1 - t^2) w(t), x)| > y/3\}.$$

Since $A_k \subset A_k^{(1)} \cup A_k^{(2)} \cup A_k^{(3)}$, then

$$A \subset \left[\bigcup_{k=2}^{\infty} (A_k^{(1)} \cup A_k^{(3)})\right] \cup \left[\bigcup_{k=2}^{\infty} A_k^{(2)}\right].$$

Therefore, in order to prove (3.3), it is enough to show that

$$y^p \int_{\bigcup_{k=2}^{\infty} (A_k^{(1)} \cup A_k^{(3)})} w(x)\, dx \leqslant C \int_E w(x)\, dx \tag{3.4}$$

and

$$\sum_{k=2}^{\infty} y^p \int_{A_k^{(2)}} w(x)\, dx \leqslant C \int_E w(x)\, dx. \tag{3.5}$$

We start with $A_k^{(1)}$. Since $x \in I_k$ and $t \in E_{11}^{(k)}$, then $1-t \geqslant x-t \geqslant 2^{-1}(1-t)$. By using this, (3.1), and $(\alpha+1)/q-(2\alpha+3)/4 \leqslant 0$, we obtain

$$\begin{aligned} &|(1-x^2)^{-1/4}\, w(x)^{-1/2}\, H(\aleph_1^{(k)}(t)\, q_{n-1}(t)\,(1-t^2)\, w(t), x)| \\ &\quad \leqslant C(1-x)^{-\alpha/2-1/4} \int_{-1}^{1} \aleph_1^{(k)}(t)\,(1-t)^{-\alpha/2-3/4}\, w(t)\, dt \\ &\quad \leqslant C(1-x)^{-(\alpha+1)/p} \int_{-1}^{1} \aleph_1^{(k)}(t)\,(1-t)^{-(\alpha+1)/q}\, w(t)\, dt. \qquad (3.6) \end{aligned}$$

By Hölder's inequality for Lorentz spaces we have

$$\begin{aligned} &\int_{-1}^{1} \aleph_1^{(k)}(t)\,(1-t)^{-(\alpha+1)/q}\, w(t)\, dt \\ &\quad = \|\aleph_1^{(k)}(t)\,(1-t)^{-(\alpha+1)/q}\|_{(1,1),w} \\ &\quad \leqslant C\|\aleph_1^{(k)}(t)\|_{(p,1),w}\, \|(1-t)^{-(\alpha+1)/q}\|_{(q,\infty),w}. \qquad (3.7) \end{aligned}$$

Also

$$\begin{aligned} &\|\aleph_1^{(k)}(t)\|_{(p,1),w} \leqslant C\|\aleph_1^{(k)}(t)\|^*_{(p,1),w} \\ &\quad = C\left(\int_{E_{11}^{(k)}} w(x)\, dx\right)^{1/p} \leqslant C\left(\int_E w(x)\, dx\right)^{1/p}. \qquad (3.8) \end{aligned}$$

Finally, we get

$$\|(1-t)^{-(\alpha+1)/q}\|_{(q,\infty),w} = \sup_{y>0} y^q \int_{|(1-t)^{-(\alpha+1)/q}|>y} w(t)\, dt < \infty. \qquad (3.9)$$

Using (3.6), (3.7), (3.8), and (3.9) in turn we obtain for $x \in A_k^{(1)}$

$$\begin{aligned} &|(1-x^2)^{-1/4}\, w(x)^{-1/2}\, H(\aleph_1^{(k)}(t)\, q_{n-1}(t)\,(1-t^2)\, w(t), x)| \\ &\quad \leqslant C(1-x)^{-(\alpha+1)/p}\left(\int_E w(x)\, dx\right)^{1/p}. \qquad (3.10) \end{aligned}$$

This argument can be made analogously for $A_k^{(3)}$ $(k \geqslant 2)$ and we have the same estimates. Thus

$$A_k^{(1)} \cup A_k^{(3)} \subset \left\{x \in (0,1) : C(1-x)^{-(\alpha+1)/p}\left(\int_E w(x)\, dx\right)^{1/p} > y\right\},$$

and as the same is true for $\bigcup_{k=2}^{\infty}(A_k^{(1)} \cup A_k^{(3)})$, (3.4) follows.

Let us see now what happens to $A_k^{(2)}$, $k \geqslant 2$. As $x \in I_k$ we can remove the term $(1+x)^{-\beta/2-1/4}$, and so

$$A_k^{(2)} \subset \{x : 2^{-k-1} < 1-x < 2^{-k},\ w(x)^{-1/p} \times |H(\aleph_2^{(k)}(t)\, q_{n-1}(t)\,(1-t^2)\, w(t), x)| > C2^{-k(\alpha/2+1/4)+\alpha k/p} y\}.$$

Then, by using (3.1) and M. Riesz's Theorem and the location of $E_{12}^{(k)}$, we obtain

$$y^p \int_{A_k^{(2)}} w(x)\, dx \leqslant C \int_{E_{12}^{(k)}} w(x)\, dx.$$

As x may belong to at most three intervals of the form $\{1-2^{-k+1} \leqslant x < 1-2^{-k-2}\}$ and the $A_k^{(2)}$ are non-overlapping intervals, we have

$$y^p \int_{\bigcup_{k=2}^{\infty} A_k^{(2)}} w(x)\, dx = y^p \sum_{k=2}^{\infty} \int_{A_k^{(2)}} w(x)\, dx \leqslant C \sum_{k=2}^{\infty} \int_{E_{12}^{(k)}} w(x)\, dx \leqslant C_1 \int_E w(x)\, dx,$$

and the theorem is shown.

Remarks. (1) As the referee has pointed out to us, recently L. Colzani, S. Giulini, and G. Travaglini [25] have considered weak type boundedness of polyhedral partial sum operators on certain compact Lie groups, proving that weak type fails in the lower endpoint. As a consequence, results for some Jacobi–Fourier series can be obtained from their work.

(2) We also have a proof of Theorem 3 when either α or β is less than $-1/2$. It can be done by modifying the proof slightly and using non-uniform estimates of Jacobi polynomials (see [11]).

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